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Non-monotonic curvature-dependent propagation

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Some flames and other kinds of interface appear to have normal propagation speeds that depend non-monotonically on curvature. The gradient of this dependence determines whether disturbances about the interface grow antidiffusively or decay diffusively and, in general, windows of antidiffusive behaviour may appear over finite ranges of curvature, which carry the 'wrong' gradient of propagation speed. A number of peculiarities arise. For instance, an interface can, under appropriate conditions, propagate subject to an antidiffusive instability for an infinite time but remain well-posed with respect to initial conditions for all time! It is not well known, but is shown here, that the linear diffusion equation $u_t = Du_{xx}$, with time-dependent coefficient $D(t)$, does not become ill-posed with respect to initial conditions when its diffusion coefficient becomes negative, but when the integral of the diffusion coefficient with respect to time becomes negative. Structural well-posedness is examined and is shown to depend on the spectrum of any 'noise' that may affect the propagation of the interface. Steadily propagating solutions are investigated; among them solutions can be identified that appear to involve a 'phase separation' in curvature space, on arbitrarily fine scales, between different solutions of fixed form. From a broader modelling perspective, antidiffusively unstable behaviour needs to be regularized in some way. Regularization would also provide a lower limit to the appearance of fine structure. The Kuramoto–Sivashinsky equation suggests a regularized model, which effectively adds a normal-propagation nonlinearity to the Cahn–Hilliard equation. This reinforces the notion that non-monotonicity in a speed–curvature relationship may be associated with 'phase separation' in curvature space. Understanding better the nature of this association in the presence of normal propagation, and any other features of the dynamics of the interface, requires further (numerical as well as analytical) investigation.

Keywords: interface dynamics; instability; phase separation; flames; meandering rivers

1. Introduction

Differences between the diffusivities of heat and a lean reactant are known to either stabilize or destabilize flames, depending on the Lewis number of the deficient reactant. In this way, the shape of a lean hydrogen flame is found to develop an antidiffusive instability, which can be identified with the negative Markstein length $\mathcal{L}$ found for such flames. $\mathcal{L}$ is defined such that the laminar flame speed V depends linearly on mean curvature κ in the manner $V \approx V_0(1+\kappa\mathcal{L})$, at least for small curvatures (Markstein 1951). In its linearized limit, a nearly flat upwardly propagating flame satisfying this law for flame speed would be described by the equation $y_t/V_0 \approx 1+\mathcal{L}y_{xx}$, which is clearly antidiffusive in character when $\mathcal{L}$ is negative. Such an equation has the property that arbitrarily small changes in initial conditions can lead to arbitrarily large and rapid changes in its solution arbitrarily quickly, so being ill-posed with respect to general initial conditions. Lean reactants that are heavy enough do not develop such an instability and are found to possess a positive Markstein length, diffusively stabilizing the shape of a propagating flame.

Recently, Kerr & Dold (2000) examined mixtures near stoichiometry and found a broader class of curvature dependence in the propagation speed, this dependence reducing to that of lean reactants as departure from stoichiometry is increased. For lean reactants with non-unit Lewis number, it is mainly the dependence of flame temperature on curvature, and, of course, the strong exponential sensitivity of reaction rate to temperature that determines the relationship between flame speed and curvature. Because the flame temperature is found to depend monotonically on curvature, so too is the flame speed. However, near stoichiometry there is an additional algebraic factor that affects the reaction rate, namely the residual concentration of the locally rich reactant in the flame's reaction zone. This also depends on curvature when the Lewis numbers are not unity. Kerr & Dold (2000) show that combinations of Lewis numbers can be found for which non-monotonic curvature dependence arises for the flame speed. In particular, this happens when the algebraic dependence dominates at weaker curvatures, while an opposing exponential dependence dominates at larger curvatures. This effect was not present in the analysis of Aldushin *et al*. (1995) in their stability analysis of nearly stoichiometric flames because they admitted only temperature dependence, through adopting zero-order concentration dependence in their model for reaction rate.

These studies employed a constant density, or small heat release approximation, through which fluid-dynamical aspects of the problem are eliminated. In real flames, thermal expansion is certainly very important, but little is known to date about the implications of non-monotonic curvature dependence in the propagation of an interface, whether or not other influences are present. It is therefore worthwhile studying its effects in isolation before attempting to combine it with fluid-dynamical influences. In some approximations, shocks and reactive shocks are believed to propagate at curvature-dependent speeds (Stewart & Bdzil 1988). Chemical systems, or models of chemical systems, may well exist in which that dependence becomes non-monotonic. Other physical systems, such as the movement of meandering rivers (Seminara & Tubino 1992), exhibit curvature-dependent propagation which may be non-monotonic, and therefore the present discussion has a much wider potential applicability.

Non-monotonic dependence of propagation speed on curvature poses some challenging mathematical problems, whether this is applied to flames or to any other form of interface. The literature based on movement by mean curvature misses these aspects by generally considering a stabilizing dependence on mean curvature. Sethian (1985) examined an interface in two dimensions, finding an amazing degree of simplicity in the evolution of the total variation (integral of the modulus of curvature) of an interface propagating with any form of curvature-dependent speed, as will be described in more detail later. His most striking result relies on assuming continuity

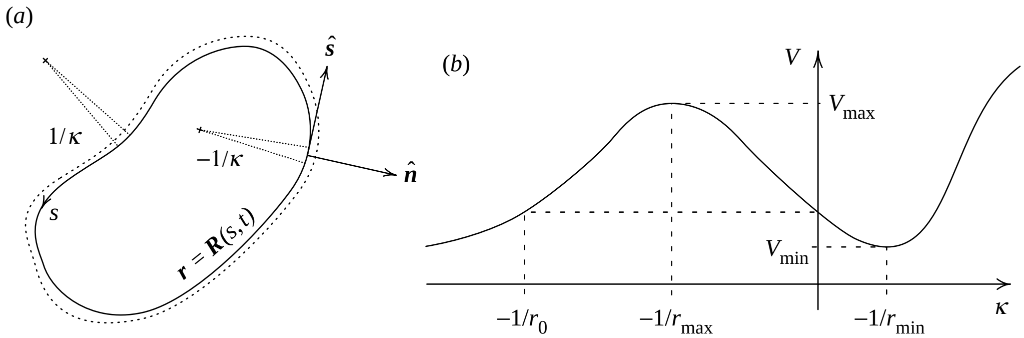

Figure 1. (*a*) A propagating interface illustrating the sign convention in the definition of curvature κ and a sample movement of the interface from the solid line to the dashed line over some time-interval. (*b*) A possible speed–curvature relationship.

of curvature. On closer examination, this assumption is unreliable, in the sense of being ill-posed for wide classes of non-monotonic speed–curvature relationships; it can be violated arbitrarily quickly as a result of arbitrarily small changes in initial conditions, as we shall see.

On the other hand, non-monotonicity has surprising features that reveal little-known properties of diffusion equations in which the coefficient of diffusion may change sign. Astonishingly, perhaps, in the exact (or noise-free) formulation of the problem with non-monotonic curvature dependence, some interfaces can propagate subject to an antidiffusive linear instability for an infinite length of time and still remain well-posed with respect to initial conditions for all time! Structural ill-posedness (the effect of weak noise in the formulation) is another question that is explored below. This part of the analysis is mostly based on linearization around circular interfaces, which is sufficient and indeed central to revealing the principal features of linear diffusivity and antidiffusivity in the dynamics. The discussion is centred mainly on one form of non-monotonic speed–curvature law, although the lessons learned in the analysis are easily generalized to much broader classes.

We also examine steadily propagating nonlinear solutions, of which there are broad classes with piecewise continuous curvature. There appears to be something akin to phase separation, in curvature space, of the interface in these solutions, in which the phase separation may occur on arbitrarily fine scales. In general, to be uniformly well-posed mathematically, antidiffusive models need to be supplemented by higher-order regularizing terms, as for example in the Kuramoto–Sivashinsky equation. For flames this would reflect the fact that a flame has a finite thickness, providing a lower limit to the length-scale of possible unstably growing disturbances. Considering such higher-order effects, in connection with the Kuramoto–Sivashinsky equation, leads to a natural generalization of the Kuramoto–Sivashinsky equation in which both non-monotonic curvature-dependent propagation and a fourth-order regularizing effect, the diffusion of curvature, are included. This equation is also found to be a version of the Cahn–Hilliard equation, extended through inclusion of nonlinearities arising from normal propagation. Discussion of such higher-order effects is relegated to an appendix in order to focus attention on the potential for regularization provided by non-monotonicity in the speed–curvature law alone.

2. Model

In two dimensions,† an interface may be parametrized in terms of arclength s and time t by a vector-valued function of the form $\boldsymbol{r} = \boldsymbol{R}(s,t)$ with tangent unit vector $\hat{\boldsymbol{s}} = \boldsymbol{R}_s$ and normal unit vector $\hat{\boldsymbol{n}}$ (chosen to point in the direction of propagation) as illustrated in figure 1a. In order to be consistent with convention in the combustion-theory literature we define curvature by $\kappa = \hat{\boldsymbol{s}}_s \cdot \hat{\boldsymbol{n}} = -\hat{\boldsymbol{n}}_s \cdot \hat{\boldsymbol{s}}$, which sets κ to be negative when the interface is convex as viewed from the direction of propagation (see figure 1a). If the interface propagates in the normal direction $\hat{\boldsymbol{n}}$ at the curvature-dependent speed $V(\kappa)$, then the evolution of the interface is determined by

$$\boldsymbol{R}_t \cdot \hat{\boldsymbol{n}} = V(\kappa), \tag{2.1}$$

starting with any suitable continuous initial shape of the interface $\boldsymbol{R}(s,0) = \boldsymbol{R}_0(s)$. In this paper we will mostly consider a speed–curvature law of the form illustrated in figure 1b, having $V > 0$ for all values of κ, one local maximum $V_{\max} = V(-1/r_{\max})$ and one local minimum $V_{\min} = V(1/r_{\min})$ at a negative and a positive curvature respectively, V decreasing as $\kappa \to -\infty$, and V increasing without bound (or at least above the local maximum value) as $\kappa \to +\infty$. It is also worth identifying the negative curvature $-1/r_0$ at which the speed is the same as that for a flat interface, $V(-1/r_0) = V(0)$.

This form of law is prompted by the analysis of Kerr & Dold, but many other forms could be considered in exactly the same way as is done below. For the overall shape of the law that we adopt, it is not necessary to provide a more exact formulation, because many of the results obtained here are general enough to be extended readily to wide classes of speed–curvature relationship.

An outwardly propagating circular interface has a negative curvature that decreases in magnitude (increases positively in value) over time, while an inwardly propagating interface has a positive and increasing curvature. If the radius is $r = a(t)$, then the two cases satisfy

$$\frac{\mathrm{d}a}{\mathrm{d}t} = V(-1/a) \implies t = \int_{a_0}^{a} \frac{\mathrm{d}a}{V(-1/a)} \tag{2.2}$$

and

$$\frac{\mathrm{d}a}{\mathrm{d}t} = -V(1/a) \implies t = \int_{a}^{a_0} \frac{\mathrm{d}a}{V(1/a)}, \tag{2.3}$$

respectively, where a_0 is the initial radius. The first formulation (2.2) includes the latter if one considers inward propagation to correspond to a negative radius.

In polar coordinates, if the interface lies at $r = a(t) + u(\theta, t)$, then it satisfies the equation

$$\frac{\mathrm{d}a/\mathrm{d}t + u_t}{(1 + u_\theta^2/r^2)^{1/2}} = \pm V\left(\pm\frac{r u_{\theta\theta} - 2u_\theta^2 - r^2}{r^3(1 + u_\theta^2/r^2)^{3/2}}\right), \tag{2.4}$$

with the positive or negative signs being taken for outwardly or inwardly propagating interfaces, respectively. From this it can be noted that the curvature (the argument

† Much of the analysis easily extends to three dimensions without revealing any major difference from a simpler two-dimensional study, at least in linearized cases. Nonlinear solutions in three dimensions involve more complexity but are conceptually a straightforward extension.

of the function V) linearizes in the manner $\kappa = \mp 1/a \pm (u + u_{\theta\theta})/a^2 + O(u^2)$ when u is small.

3. Small deviations from a circular interface

Considering the deviation u from a perfectly circular outwardly propagating interface to be small, the model (2.4) takes on the asymptotic form

$$u_t = \frac{V'(-1/a)}{a^2}(u_{\theta\theta} + u) \quad \text{with } \frac{\mathrm{d}a}{\mathrm{d}t} = V(-1/a). \tag{3.1}$$

If we were to consider inward propagation, the equation for $u(\theta, t)$ would be the same apart from a change in sign from $-1/a$ to $1/a$, both representing none other than the leading-order value of the curvature κ, while the equation for $a(t)$ would change to its form in equation (2.3). Alternatively, as mentioned before, we could simply take a to be negative for inward propagation. In either case, the equation for u is, essentially, a diffusion equation having a non-constant diffusion coefficient, $\kappa^2 V'(\kappa)$ to leading order, whatever the sign of κ. Written in terms of arclength, either equation would become $u_t = V'(\kappa)(u_{ss} + \kappa^2 u)$ to leading order, with diffusion coefficient $V'(\kappa)$, but it is sufficient for the present discussion to focus attention initially on equation (3.1).

(*a*) *Well-posedness*

The main feature of equation (3.1) is that the 'diffusion coefficient' can be negative, namely wherever the function $V(\kappa)$ has a negative gradient. Such a sign is normally associated with a diffusion problem being ill-posed, but we should note that $V'(\kappa)$ is not universally negative for the speed–curvature law we are considering. Indeed it is only negative in a finite window of curvature space, namely $-1/r_{\max} < \kappa < 1/r_{\min}$, and this has some significant consequences.

The transformation to a pseudo-time or 'age' variable $\tau(t)$, which we define as the time-integral of the diffusion coefficient, allows equation (3.1) to be rewritten as

$$u_\tau = u_{\theta\theta} + u \quad \text{with } \tau = \int_0^t V'(-1/a)\frac{\mathrm{d}t}{a^2}, \tag{3.2}$$

which is a diffusion equation having a constant positive coefficient of precisely unity. Such an equation is well-posed with respect to initial conditions for all positive values of τ. Indeed since the interface must be continuous, $u(\theta, 0)$ is continuous and possesses a Fourier representation

$$u(\theta, 0) = \sum_{k=-\infty}^{\infty} A_k \mathrm{e}^{\mathrm{i}k\theta}$$

(with $A_k = A^*_{-k}$ since u is real). Separation of variables therefore leads to a unique solution,

$$u = \sum_{k=-\infty}^{\infty} \mathrm{e}^{(1-k^2)\tau} A_k \mathrm{e}^{\mathrm{i}k\theta}, \tag{3.3}$$

provided only that the sum converges. Clearly, it converges for all positive values of τ, but to converge for any negative value of τ the coefficients A_k would need to decay in a Gaussian manner with wavenumber, such that $|e^{k^2|\tau|}A_k|$ would be bounded for all wavenumbers and decay suitably rapidly as $|k| \to \infty$. Tiny changes in initial condition (for example, adding ϵ/k^{n+1} to A_k, for any small ϵ and $n \geqslant 1$, which corresponds to introducing arbitrarily small discontinuities in only the nth arclength derivative of the interface) can violate this requirement and render the sum divergent at any negative pseudo-time, making the problem ill-posed for any 'age' $\tau(t) < 0$.

To complete the solution (3.3) it remains only to evaluate τ. It turns out that this can be done exactly in terms of $a(t)$, since

$$\tau = \int_{a_0}^{a} \frac{V'(-a^{-1})}{V(-a^{-1})} \frac{\mathrm{d}a}{a^2} = \int_{-1/a_0}^{-1/a} \frac{V'(\kappa)}{V(\kappa)} \, \mathrm{d}\kappa = \ln \frac{V(-1/a)}{V(-1/a_0)}. \tag{3.4}$$

Clearly, τ is positive as long as the propagation speed at radius a exceeds the initial speed at radius a_0. If the initial radius is small enough, less than r_0 for the speed–curvature law sketched in figure 1*b*, then this is true for all time; and so the problem is then well-posed for all time in spite of the 'diffusion coefficient' being negative for an unlimited time!

Inherent in this result is the following little-known *theorem* of diffusion or 'heat' equations (which the authors have not seen quoted before).

Theorem 3.1. *The linear diffusion equation $u_t = Du_{xx}$, with time-dependent coefficient $D(t)$, becomes ill-posed with respect to initial conditions when the integral of the diffusion coefficient with respect to time becomes negative (which is not necessarily when the diffusion coefficient itself becomes negative).*

In fact, if $D(t)$ starts off being positive, then by the time that D changes sign the coefficients in the Fourier transform of the solution will have become Gaussian due to the smoothing effect created while D was positive. This allows the solution to survive an equivalent degree of negative diffusivity as discussed above. As equation (3.4) shows, this can itself provide a remarkable and very simple regularizing effect for propagating interfaces.

(*b*) *The effect of 'noise'*

The results above apply if the interface propagates exactly according to the model (2.1). In physically relevant situations small perturbations might well be induced at any time, perhaps arising because of 'noise' or spatial non-uniformities in the medium through which the interface moves. In order to examine their effect we can consider the introduction of a non-homogeneous term to equation (3.1) in the manner

$$u_t = \frac{\mathrm{d}\tau}{\mathrm{d}t}(u_{\theta\theta} + u) + f \quad \text{with } f(\theta, t) = \sum_{k=-\infty}^{\infty} f_k(t) e^{ik\theta}. \tag{3.5}$$

This equation has the particular integral

$$u_p(\theta, t) = \sum_{k=-\infty}^{\infty} u_k(t) e^{ik\theta} \quad \text{with } u_k(t) = e^{(1-k^2)\tau(t)} \int_0^t e^{(k^2-1)\tau(\zeta)} f_k(\zeta) \, \mathrm{d}\zeta, \tag{3.6}$$

which completes the solution of (3.5) when it is added to the homogeneous solution in (3.3), provided only that the sums converge.

When k is large, the kernel of the integral in (3.6) is dominated by behaviour of the exponential near the maximum value of τ (where $V'(-1/a)$ changes sign). Taking the behaviour of τ near its maximum to be quadratic in the manner

$$\tau = \tau_0 - \beta(t-t_0)^2 + O(t-t_0)^3 \quad \text{having } \beta = -\frac{V''(-r_{\max}^{-1})}{2r_{\max}^4} V(-r_{\max}^{-1}) > 0. \qquad (3.7)$$

Watson's lemma leads to the asymptotic estimate for u_k when $t > t_0 + O(k)$, as $|k| \to \infty$:

$$u_k \sim \mathrm{e}^{(k^2-1)(\tau_0-\tau)} \frac{\alpha_k}{|k|} \quad \text{with } \alpha_k = \frac{1}{\sqrt{\beta}} \int_{-\infty}^{\infty} \mathrm{e}^{-\nu^2} f_k\left(t_0 + \frac{\nu}{|k|\sqrt{\beta}}\right) \mathrm{d}\nu. \qquad (3.8)$$

If $f_k(t)$ varies slowly on the fast time-scale $\nu = |k|\beta^{1/2}(t-t_0)$, when $\nu = O(1)$, then, as $|k| \to \infty$, $\alpha_k \sim (\pi/\beta)^{1/2} f(t_0)$. More generally, the constant α_k picks out a Gaussian-weighted average level of noise in the mode f_k, in a narrow band around the moment when $\mathrm{d}\tau/\mathrm{d}t$, or $V'(-1/a)$, changes sign, and is relatively independent of f_k at other times.

As previously, the sum in (3.6) only converges for $t > t_0$ if $|\mathrm{e}^{k^2|\tau_0-\tau|}\alpha_k|$ is at least bounded, which, in turn, requires the noise to decay in a Gaussian manner with wavenumber. This may be a severe restriction in some cases if noise were to be present and to have an algebraic spectrum, as in the inertial range of turbulence, for example, or in numerical simulations where rounding and truncation errors would always be present. In such cases, it seems that a strong structural instability in the model (2.1) could take over arbitrarily quickly. In practice, any algebraic tail-off in a noise spectrum should ultimately give way to Gaussian, or superexponential, decay at small enough scales, such as below the Kolmogorov length in the case of fluid-dynamical turbulence. This should limit the rapidity with which the effects of noise would become significant.

4. Total variation

In examining what might happen for larger amplitude fully nonlinear evolutions it is useful to recall the work of Sethian (1985, 1996), who obtained a remarkable result concerning the evolution of the total variation $C(t)$, defined as the integral of the absolute value of curvature over the entire interface. That is,

$$C = \int_0^L |\kappa| \, \mathrm{d}s, \qquad (4.1)$$

where $L(t)$ is the total arclength of the interface. For any simple closed curve in which κ does not change sign (therefore making it uniformly negative) the value of C is exactly 2π. If κ does change sign, then C is larger than 2π. The evolution of C is an indication of how the interface might be distorting or smoothing itself out as it propagates.

Using the relations, $\hat{\boldsymbol{s}}_s = \kappa\hat{\boldsymbol{n}}$, $\hat{\boldsymbol{n}}_s = -\kappa\hat{\boldsymbol{s}}$ and $\hat{\boldsymbol{n}}_t \cdot \hat{\boldsymbol{n}} = \hat{\boldsymbol{s}}_t \cdot \hat{\boldsymbol{s}} = 0$, the model (2.1) can be differentiated to give

$$\kappa_t = V_{ss} + \kappa^2 V + \boldsymbol{R}_t \cdot \hat{\boldsymbol{s}}\kappa_s = (V_s + \kappa \boldsymbol{R}_t \cdot \hat{\boldsymbol{s}})_s. \qquad (4.2)$$

Splitting the integral for C into integrals over intervals where κ is either positive or negative, it takes the form

$$C = \sum_{n=1}^{N} C_n \quad \text{with } C_n = \frac{\kappa}{|\kappa|} \int_{s_n}^{s_{n+1}} \kappa \, \mathrm{d}s, \tag{4.3}$$

where $s_{N+1} = s_1 + L$ and κ has one sign only, represented by $\kappa/|\kappa|$, or is zero, between successive values of s_n. If we take curvature to be continuous, so that κ is zero where it changes sign, then

$$\begin{aligned} \frac{\mathrm{d}C_n}{\mathrm{d}t} &= \frac{\kappa}{|\kappa|} \int_{s_n}^{s_{n+1}} \kappa_t \, \mathrm{d}s \\ &= \frac{\kappa}{|\kappa|} [V_s + \kappa \boldsymbol{R}_t \cdot \hat{\boldsymbol{s}}]_{s_n}^{s_{n+1}} \\ &= -V'(0)|\kappa_s(s_{n+1}, t) - \kappa_s(s_n, t)|, \end{aligned} \tag{4.4}$$

the final part of which arises because of the sign restriction that κ_s must have when entering or leaving any one interval. To obtain $\mathrm{d}C/\mathrm{d}t$, these derivatives must be added and account taken of the appearance or disappearance of intervals through κ changing sign locally. The latter possibility was not considered in Sethian's (1985) paper, but if κ is continuously differentiable, κ_s is zero when an interval appears or disappears and this aspect makes no contribution to $\mathrm{d}C/\mathrm{d}t$.

It is now readily seen that, subject to certain assumptions, any increase or decrease of $C(t)$ is determined entirely by the gradient of the speed–curvature relationship at zero curvature, $V'(0)$. If $V'(0) > 0$, then $\mathrm{d}C/\mathrm{d}t \leqslant 0$; if $V'(0) < 0$, then $\mathrm{d}C/\mathrm{d}t \geqslant 0$; and if $V'(0) = 0$, then C is a constant. If this remarkable conclusion should hold in general, then global features of interfaces would be greatly simplified. Any evolution resulting from the antidiffusive behaviour we have identified for any $V'(\kappa) < 0$, would be strictly limited by only the local properties of the speed–curvature relationship near zero curvature. For example, if κ were uniformly negative to start with, it would never be able to change sign if $V'(0) \geqslant 0$ regardless of the variation of the function $V(\kappa)$ elsewhere. In view of the strong nature of the instabilities identified, such a restriction seems unlikely to hold for general functions $V(\kappa)$.

On the other hand, foremost amongst the assumptions leading to Sethian's (1985, 1996) conclusion is the continuity of curvature. Other assumptions might possibly be dispensable, but it is clear that the arguments contained in the derivation (4.4) are easily violated if κ changes sign discontinuously with arclength.

Examples can be constructed that do not maintain Sethian's (1985, 1996) prediction, even for interfaces that initially do have continuous and twice differentiable curvature. Consider the interface sketched in figure 2*a* and the propagation law sketched in figure 2*b*. This law has a zero gradient at $\kappa = 0$ so that C should stay constant. However, for negative curvatures $\kappa < \kappa_1 < 0$, it has a constant propagation speed V_1. Around $\kappa = 0$, in the interval $\kappa_2 < \kappa < \kappa_3$ that straddles the origin, the speed is $V_0 < V_1$. For larger values of κ we can (for example) take V to be increasing but this makes no difference. The initial shape in figure 2*a* can be chosen to be almost a polygon, having regions of high negative curvature ($\kappa < \kappa_1$) and regions of nearly zero curvature ($\kappa_2 < \kappa \leqslant 0$) joined by any smoothly varying negative curvature. Clearly, $C = 2\pi$ initially. However, the regions of near-zero curvature propagate at

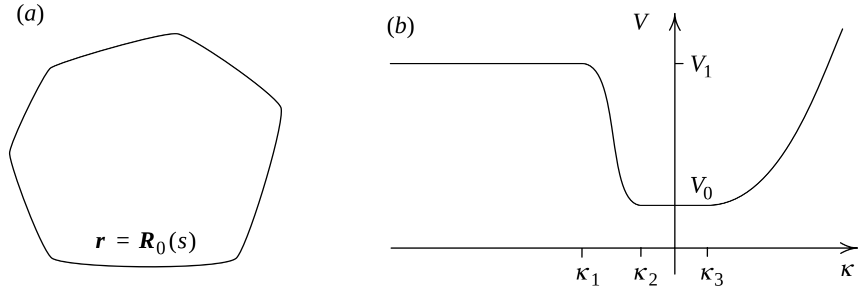

Figure 2. (*a*) A propagating interface. (*b*) A possible speed–curvature relationship. Under these conditions, the interface will undergo changes in the sign of curvature.

speeds lower than the regions of large, negative curvature. The curve can easily be chosen such that this discrepancy in propagation speeds brings about an inversion in curvature in arbitrarily short times, causing C to increase above 2π.

This amounts to a proof that the evolution cannot maintain the degree of smoothness in curvature required for Sethian's (1985, 1996) results to remain valid. Moreover, because any smooth curve can be approximated arbitrarily closely by another smooth shape that approaches a high-order polygon, arbitrarily small, although still smooth, disturbances to any initial shape can cause Sethian's (1985, 1996) result to be violated arbitrarily quickly. The actual evolution of total variation in any one example must therefore depend on the nature of the speed–curvature law over much more of its range than simply the origin.

It is likely that Sethian's (1985, 1996) results would be maintained for strictly monotonic speed–curvature laws, by which $V'(\kappa)$ remains either strictly positive or strictly negative for all accessible curvatures κ. In other cases, the example above shows that his conclusions are easily violated. In doing so, of course, the assumptions underlying Sethian's (1985, 1996) deduction must be violated and in particular it is likely that curvature becomes discontinuous.

5. Discontinuous solutions: propagation without change of shape

The appearance of discontinuous solutions can be demonstrated by considering interfaces that propagate without change of shape. If an interface propagates in the y-direction at a speed c and follows the path $y = ct + v(x)$, then the shape of the interface is determined by

$$\frac{c}{(1+v'^2)^{1/2}} = V\left(\frac{v''}{(1+v'^2)^{3/2}}\right). \tag{5.1}$$

Any solution can be characterized (modulo translations in x and y) by the initial conditions $v(0) = v'(0) = 0$ and equation (5.1) makes it clear both that solutions would then be symmetric about $x = 0$ and that $V(v'') = c$ at $x = 0$. Since v'^2 increases away from $v' = 0$, the second derivative v'' must vary such that V decreases away from $x = 0$.

In general, for the speed–curvature law of figure 1*b*, with $V_{\min} < c < V_{\max}$, there are three possible solutions determined by the three intersections of the curve with the line $V = c$, as sketched in figure 3*a*. By labelling these I, II and III from left to right, the qualitative features of each of these solutions can be described as follows.

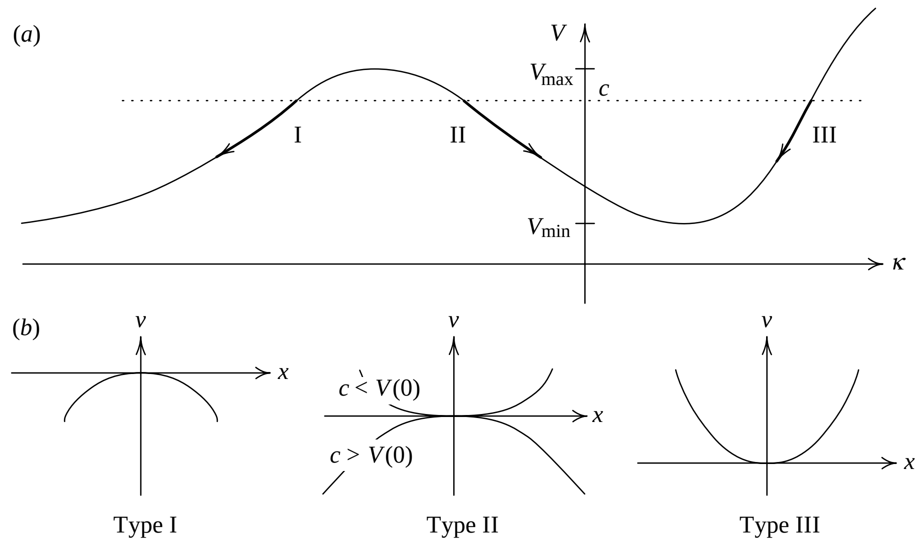

Figure 3. (a) Intersections between the propagation law of figure 1 with the line $V = c$, leading to three types of steadily propagating solution (shown uppermost). (b) Each of the different qualitative shapes of steady solution $v(x)$ that can arise.

Type I. These solutions have negative curvature, which increases negatively away from their maximum point (where $v' = 0$). They must be bounded above by a semicircle of curvature equal to $v''(0)$ and so they have bounded support approaching infinite negative curvature at their endpoints. Along the lines of the analysis in § 3, these solutions would be stable to small-scale disturbances, since $V'(\kappa) > 0$ for all curvatures found along the curve.

Type II. These solutions have three alternative forms depending on whether $c > V(0)$, $c < V(0)$ or $c = V(0)$. In the latter case they are simply constant, $v \equiv 0$. These curves are all unstable to small-scale disturbances, since $V'(\kappa) < 0$. For $c > V(0)$ curvature is negative and increases towards zero away from a maximum value, so that the solutions have unbounded support. For $c < V(0)$ curvature is positive and increasing, but because the curvature is bounded below by the value $v''(0)$ (and the solution is therefore bounded below by a semicircle of that curvature), the solutions have bounded support with curvature approaching $1/r_{\min}$ at their endpoints.

Type III. These solutions have positive curvature, decreasing away from the minimum of $v(x)$ where $v' = 0$. However, because the curvature is bounded below by the value $1/r_{\min}$, these solutions also have bounded support and approach a curvature of $1/r_{\min}$ at their endpoints. These solutions are stable to small-scale disturbances, since $V'(\kappa) > 0$.

Each type of solution is sketched qualitatively in figure 3b.

We can note that if $V(\kappa)$ tends to zero as $\kappa \to -\infty$, and to infinity as $\kappa \to \infty$, the system cannot dynamically sustain infinite curvatures (that is, discontinuities in direction, or cusps). An excessively sharp positive curvature diminishes through propagating forwards quickly and an excessively negative curvature diminishes through being caught up by surrounding faster propagation. This being so, the tangent vector $\hat{\boldsymbol{s}}$ must be continuous even though the curvature might be discontinuous.

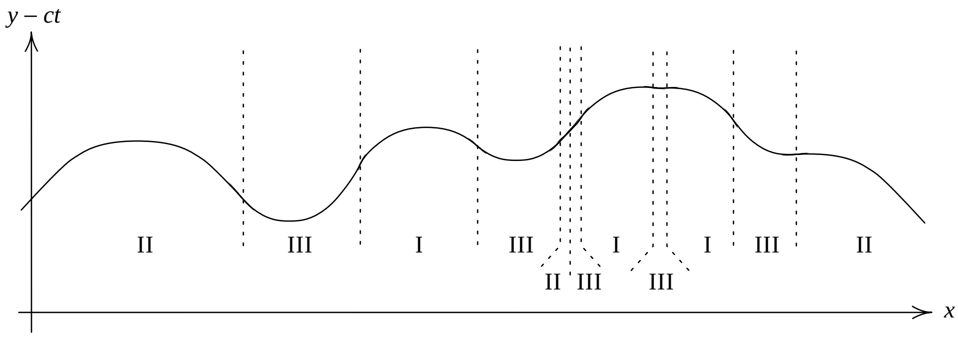

Figure 4. An amalgamation of type I, II and III solutions providing one example of an interface of fixed form propagating at speed c. Different types of solution can be connected with continuous slope on arbitrarily fine scales.

In the context of solutions propagating with constant form this means that quite complicated overall solutions can be constructed out of suitable translations of type I, II and III solutions for a given value of c. Any suitable portion of one type of solution needs only to be joined together with a suitable portion of the next at a point where both solutions share the same slope. This portion can in turn be joined to another in the same manner. At points where different types of solution meet, each has the same normal propagation speed and normal direction vector $\hat{\boldsymbol{n}}$, so that the overall solution is therefore consistent. It can be noted that there is no lower limit to the size of the interval occupied by each successive solution so that solutions can be joined together in an arbitrarily fine structure.

Other forms of behaviour become possible if $V(\kappa)$ remains finite and below $V_{\max}$ in the limit of infinite positive curvature. Cusps can then be sustained above a certain speed c. Likewise if $V(\kappa)$ were to increase towards c in a suitable manner as $\kappa \to -\infty$, sharp leading cusps could be generated. Indeed if the variation of the function $V(\kappa)$ were to be modified to generate more or fewer intersections with $V = c$, or if the locations of the maxima or minima were to be changed, then other types of solution would arise in the same kind of way. There are clearly many alternative scenarios that could mostly be examined in a similar manner.

A sketch of one possible array of type I, II and III solutions is shown in figure 4. A composite solution of this form can be regarded as having separated into different possible 'phases' where each type of solution represents a different phase. Within constraints set by the nature of the propagation law $V(\kappa)$ that controls the system, such a separation would permit the interface to propagate at speeds that are different from the propagation speed $V(0)$ of a flat interface. Interestingly, because 'phases' can be joined together on arbitrarily fine scales, it is possible to design composite solutions that are arbitrarily close to a flat interface but which propagate at a quite different speed.

6. Concluding remarks

It is clear that non-monotonicity in the speed–curvature relationship is responsible for a wide range of phenomena, including the fact that antidiffusive instability, for $V'(\kappa) < 0$, may then be restricted only to a window of curvatures, giving rise to a surprising degree of regularization. Questions of ill-posedness emerge as antidiffusive features appear. We have seen that non-monotonicity can in some circumstances provide a *complete regularization* of the linearized problem, although the system is also very sensitive to the presence of 'noise' or more deterministic forms of forcing. On the other hand, it has been demonstrated that discontinuities in curvature are likely to arise naturally, rendering irrelevant results that are based on assuming continuity of curvature. Multiple, steadily propagating forms of solution are found that can coexist on different parts of the same interface, satisfying suitable conditions of consistency where the solutions meet. This resembles a form of 'phase separation' in curvature space, on scales that can be arbitrarily fine.

While such a steadily propagating 'patchwork' solution can be constructed theoretically for a given propagation law $V(\kappa)$ and speed c, it represents only one of a wide class of possible solutions. Many more questions remain. It is not at all clear, for example, that a steady solution of this type would generally evolve from small perturbations in an unstable flat interface, and if it did so what value of c would be selected,† or how finely different phases might be distributed. It is much more likely that any such solution is itself unstable in a variety of ways and a far wider range of dynamical behaviour would then occur. Different 'phases' of different speed c might interact dynamically, probably involving coalescence towards larger scales, while perturbations to the antidiffusively unstable phase II could break these parts of the solution up into yet finer distributions of phases. Because normal propagation tends to make curvatures grow in the positive direction, negative curvatures are likely to evolve towards the antidiffusively unstable range if they are not already within it. The presence of any noise and its nature (especially its spectrum) are also very likely to play a major part in determining the dynamics.

The system is certainly strongly nonlinear and its general dynamics are still unknown. A way forward that is currently being investigated is the use of numerical simulations involving small levels of high-order regularization in the form

$$\boldsymbol{R}_t \cdot \hat{\boldsymbol{n}} = V(\kappa) - \delta^2 \nabla^2 \kappa, \tag{6.1}$$

which reduces to the model (2.1) if $\delta = 0$. As discussed in the appendix, this equation can be viewed as both a version of the Kuramoto–Sivashinsky equation, modified to include general nonlinear dependence of propagation speed on curvature, and as a generalization of the Cahn–Hilliard equation that involves additional nonlinearity due to normal propagation. The latter connection reinforces the notion of phase separation but it is not clear how strongly normal propagation would modify results. This paper has explored a number of avenues that are available for understanding the implications of non-monotonicity in a speed–curvature relationship through analytical investigation alone.

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Appendix A. Links with the Kuramoto–Sivashinsky and Cahn–Hilliard equations

Approximating the behaviour of small-amplitude disturbances from a flat constant-density flame (lean in a suitably light reactant) propagating upwards in the form

† The value $c = V_{\max}$ would seem to have some special significance.

$y = ct + \phi(x,t)$, the Kuramoto–Sivashinsky equation has the two alternative forms (Sivashinsky 1977; Frankel 1991)

$$\left.\begin{aligned} \phi_t - \tfrac{1}{2}c\phi_x^2 &= -\gamma\phi_{xx} - \delta^2\phi_{xxxx}, \\ \text{or } \boldsymbol{R}_t\cdot\hat{\boldsymbol{n}} - c &= -\gamma\kappa - \delta^2\kappa_{ss}. \end{aligned}\right\} \tag{A 1}$$

Both forms are equivalent asymptotically to second order for small values of ϕ_x if the interface approaches a flat interface propagating upwards at speed c. However, the latter form could be propagating in any direction (Frankel 1991), and for $\delta = 0$ it is equivalent to the model (2.1) combined with the linearly antidiffusive speed–curvature law $V(\kappa) = c - \gamma\kappa$.

A natural generalization of the Kuramoto–Sivashinsky equation that can take nonlinear and indeed non-monotonic curvature dependence into account is therefore

$$\boldsymbol{R}_t\cdot\hat{\boldsymbol{n}} = V(\kappa) - \delta^2\kappa_{ss}, \tag{A 2}$$

reducing to the model (2.1) when $\delta = 0$, and having the further three-dimensional extension (6.1) for $\delta > 0$. In terms of small deviations of the flame shape $\phi(x,t) = y - ct$ from a flat interface propagating upwards at speed c as before, this equation becomes

$$\phi_t - \tfrac{1}{2}c\phi_x^2 = V(\phi_{xx}) - c - \delta^2\phi_{xxxx}, \tag{A 3}$$

where the term $-\frac{1}{2}c\phi_x^2$ is a weakly nonlinear representation of normal propagation, coming from the exact form of normal propagation $\boldsymbol{R}_t\cdot\hat{\boldsymbol{n}} = (c+\phi_t)/(1+\phi_x^2)^{1/2}$ through binomial expansion of the denominator. Since $\kappa = \phi_{xx}$ to leading order, this equation can also be written as

$$\kappa_t = (V(\kappa) + \tfrac{1}{2}c\phi_x^2)_{xx} - \delta^2\kappa_{xxxx}, \tag{A 4}$$

which is a generalization of the Cahn–Hilliard equation in which the additional nonlinearity $\frac{1}{2}c(\phi_x^2)_{xx}$, or $c(\kappa^2 + \phi_x\kappa_x)$, represents the leading-order effect of normal propagation at speed c. Using equation (4.2) the effects of normal propagation can be included without approximation, in the coordinate-free version

$$\kappa_t - \boldsymbol{R}_t\cdot\hat{\boldsymbol{s}}\kappa_s = V\kappa^2 + (V(\kappa))_{ss} - \delta^2\kappa_{ssss}. \tag{A 5}$$

Without the extra terms describing normal propagation, $\boldsymbol{R}_t\cdot\hat{\boldsymbol{s}}\kappa_s$ and $V\kappa^2$, the Cahn–Hilliard equation models phase separation between values of κ where $V(\kappa)$ has maximum positive slope (Bai *et al.* 1995). This connection therefore establishes a stronger link with the notion of separation into different 'phases' of curvature, modified by the effects of normal propagation. The term $\boldsymbol{R}_t\cdot\hat{\boldsymbol{s}}\kappa_s$ of (A 5) simply describes reparametrization of the surface as the arclength variable s changes with time. Of more significance is the term $V\kappa^2$, representing the effect on curvature of normal propagation, as determined by Huygens's principle. Setting the left-hand side of (A 5) to zero and letting $\delta \to 0$ provides a nonlinear ordinary differential equation describing any 'phase' of curvature

$$V'(\kappa)\kappa_{ss} + V''(\kappa)\kappa_s^2 + V(\kappa)\kappa^2 = 0. \tag{A 6}$$

If $V(\kappa) = c$, where $\kappa_s = 0$, this equation becomes a coordinate-free generalization of equation (5.1).

In some sense δ is a measure of the 'thickness' of a flame and the term $-\delta^2\kappa_{ss}$ prevents the growth of disturbances of wavelength shorter than $O(\delta)$. For example, the small-disturbance equation (3.1) can then be generalized and written in the form

$$u_t - \tfrac{1}{2}V(\kappa)u_s^2 = V'(\kappa)(u_{ss} + \kappa^2 u) - \delta^2 u_{ssss} \tag{A 7}$$

for non-zero δ and weak normal-propagation nonlinearity $-\frac{1}{2}V(\kappa)u_s^2$. The final term then has exactly the same regularizing effect as it does in the Kuramoto–Sivashinsky equation (A 1), making the problem well-posed for any propagation law $V(\kappa)$ and any initial state, both with respect to initial conditions and the addition of 'noise' (structural well-posedness). Indeed, equation (A 7) is another 'local' form of the Kuramoto–Sivashinsky equation (A 1) describing small-scale perturbations to any propagating interface and, in particular, their stabilization at high enough wavenumbers even when $V'(\kappa) < 0$ (Sivashinsky 1977).

Solutions of the type sketched in figure 4 can be interpreted as outer asymptotic solutions of equation (A 2) in the limit as $\delta \to 0$. Inner asymptotic boundary layers would form to smooth out discontinuous jumps in curvature over arclength changes of the order of δ. Seen in this light, the analogy to phase separation in curvature space is even more pertinent, with type I and type III phases being locally comparable to diffusively stabilized Kuramoto–Sivashinsky evolutions (having $\gamma = -V'(\kappa) < 0$), while type II phases pick up some of the linearly antidiffusive dynamics of Kuramoto–Sivashinsky equations, in which $\gamma = -V'(\kappa) > 0$ locally.

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